Generalization of Karhunen-Loeve expansion : Stochastic Simulation involving Multi-Correlated Random Processes

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Abstract

We propose two different approaches generalizing the Karhunen-Loeve series expansion to model and simulate multi-correlated non-stationary stochastic processes. The **muKL** (multi-uncorrelated KL) method produces a series in terms an identical set of *uncorrelated* random variables, and **mcKL** (multi-correlated KL) relies on expansions in terms of *correlated* sets of random variables, both reflecting the cross-covariance structure of the processes. In particular, we study the accuracy and convergence rates of our series expansions and compare the results against other statistical techniques.

Introduction

Many stochastic systems of interest to engineering involve ***multiple random processes* with *mutual correlations***, for instance, earthquake ground motions, acoustic propagation and multi-scale modeling of materials [1,2] (See **Figure** 1 and 2). The effective mathematical representation of such processes is the key element for the efficient stochastic simulations.

Over the years many different techniques have been developed for a single random process. A popular one employs series expansions in terms of random variables, including Karhunen-Loéve (KL) decompositions [3,4].

For stochastic process with following mean and covariance kernel

$$\bar{f}(t) = \langle f(t;\omega)\rangle, \qquad C(t,\,s) = \langle f(t;\omega) f(s;\omega)\rangle$$

the KL expansion can be written as

$$f(t;\omega) = \bar{f}(t) + \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\phi_k(t)\xi_k(\omega)$$

where $\{\lambda_k,\,\phi_k\}$ are eigenvalue and ***orthogonal*** eigenfunction of the covariance kernel, and $\xi_k(\omega)$ are independent ***uncorrelated*** random variables.

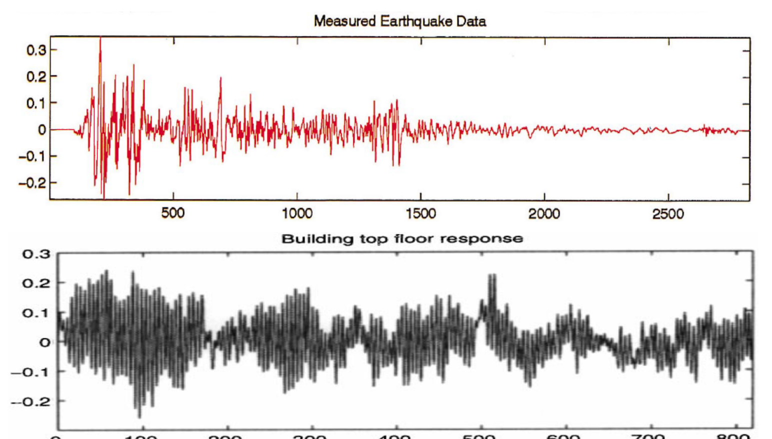


Figure 1. Earthquake data and building response as an example of correlated processes.

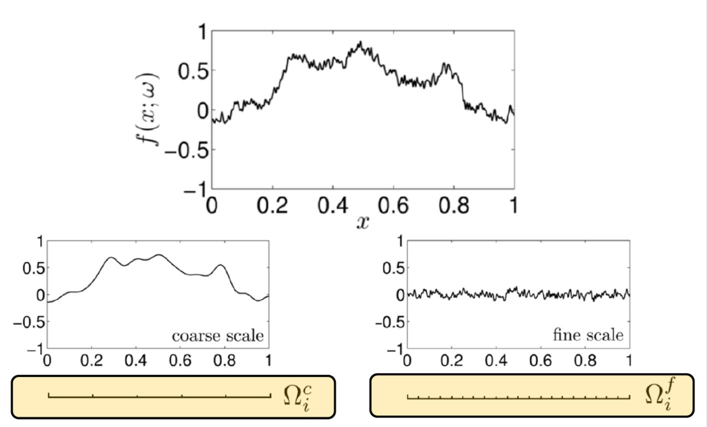


Figure 2. Correlated random process by decomposing into course and fine scales.

Methods

The effectiveness of KL expansion, for instance, optimal convergence in mean square error, is due to its *bi-orthogonality*, meaning that the eigenfunctions are *orthogonal* in L^2 and the random variables are *uncorrelated*. However, it restricts its application to single random process.

For multi-correlated random processes with following statistical structure,

$$\begin{aligned} \langle f_i(t;\omega)\rangle &= 0, && 1 \le i \le n, \\ \langle f_i(t;\omega), f_j(s;\omega)\rangle &= C_{ij}(s,\,t), && 1 \le i \le j \le n, \end{aligned}$$

We propose two algorithm by releasing either one of the bi-orthogonality [5].

multiple uncorrelated KL (muKL)	multiple correlated KL (mcKL)
Expansion with **single** set of **uncorrelated** random $\{\xi_k(\omega)\}$ var $f_i(t;\omega) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\phi_k^{(i)}(t)\xi_k(\omega)$ Not orthogonal!!	Expansion with **multiple** set of **correlated** random vari $\{\eta_k^i(\omega)\}$ $f_i(t;\omega) = \sum_{k=1}^{\infty} \sqrt{\gamma_k^i}\psi_k^i(t)\eta_k^i(\omega)$ Non-zero correlation!!

Results

1. Sample path : Both muKL and mcKL methods properly capture the correlated structure of multiple processes. See **Figure** 3.

2. Convergence : muKL method has better convergence in terms of mean square error. **Table** 1. shows the number of random variable to achieve less than 3% error in the eigenvalues.

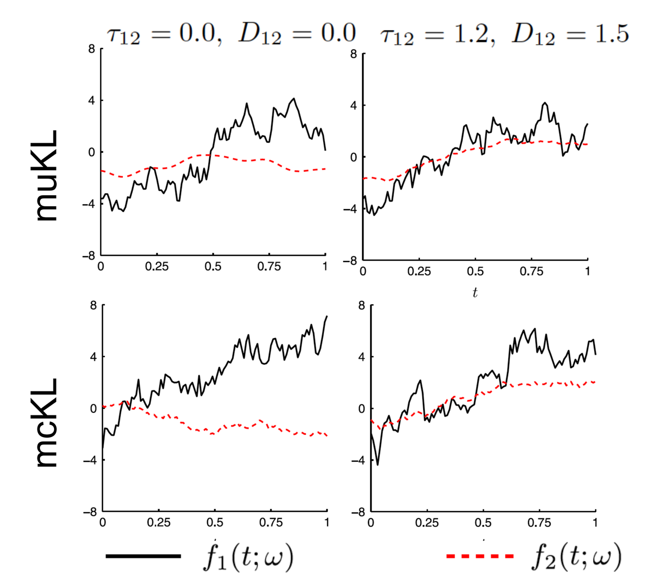


Figure 3. Sample paths of two random processes with cross-correlation length τ_{12} = 0.0 and 1.2.

τ	1.0	0.2	0.1	0.05	0.02
M (muKL)	2	5	7	13	31
(M_1, M_2) (mcKL)	(2, 1)	(3, 2)	(6, 3)	(11, 6)	(25, 13)

Table 1. Number of random variables M and M_1+M_2 to attain less than 3% mean square error for the case of correlation (τ, 2τ, 2τ).

3. Computational cost : mcKL is more efficient than muKL, due to its smaller size of eigenproblem and availability of some explicit results. See **Figure** 4.

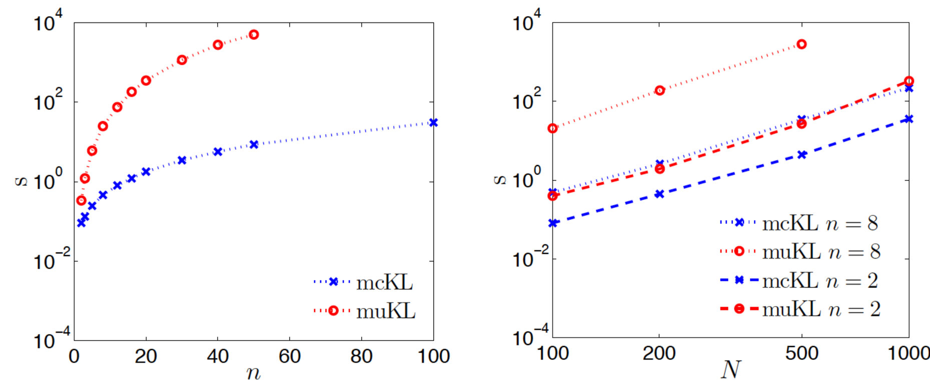


Figure 4. Computation time (in sec) versus the number random processes 'n' and the number of points in the space/time domain 'N'.

Application

Tumor concentration with treatment modeled by random processes.

$$\dot{x}(t;\omega) = G(x) + g(x) f_1(t;\omega) + f_2(t;\omega)\, x(0;\omega) = x_0(\omega)$$

$x(t;\omega)$: concentration of the tumor cell at time t
$f_1(t;\omega)$: strength of the treatment (e.g., chemotherapy, radiotherapy)
$f_2(t;\omega)$: factors that restrains the tumor cell (e.g., drugs, radiotherapy)

where random processes are mutually correlated with the following structure,

$$[\tau_1, \tau_2] = [0.5, 0.5] \quad [D_1, D_2] = [0.1, 0.1]$$

$$\begin{cases} \langle f_1(t;\omega)\rangle = 0, \quad \langle f_2(t;\omega)\rangle = 0 \\ \langle f_1(t;\omega) f_1(s;\omega)\rangle = \frac{D_1}{\tau_1} e^{-\frac{|t-s|}{\tau_1}}, \quad \langle f_2(t;\omega) f_2(s;\omega)\rangle = \frac{D_2}{\tau_2} e^{-\frac{|t-s|}{\tau_2}} \\ \langle f_1(t;\omega) f_2(s;\omega)\rangle = \frac{D_{12}}{\tau_{12}} e^{-\frac{|t-s|}{\tau_{12}}} \end{cases}$$

➔ : and τ_{12}.

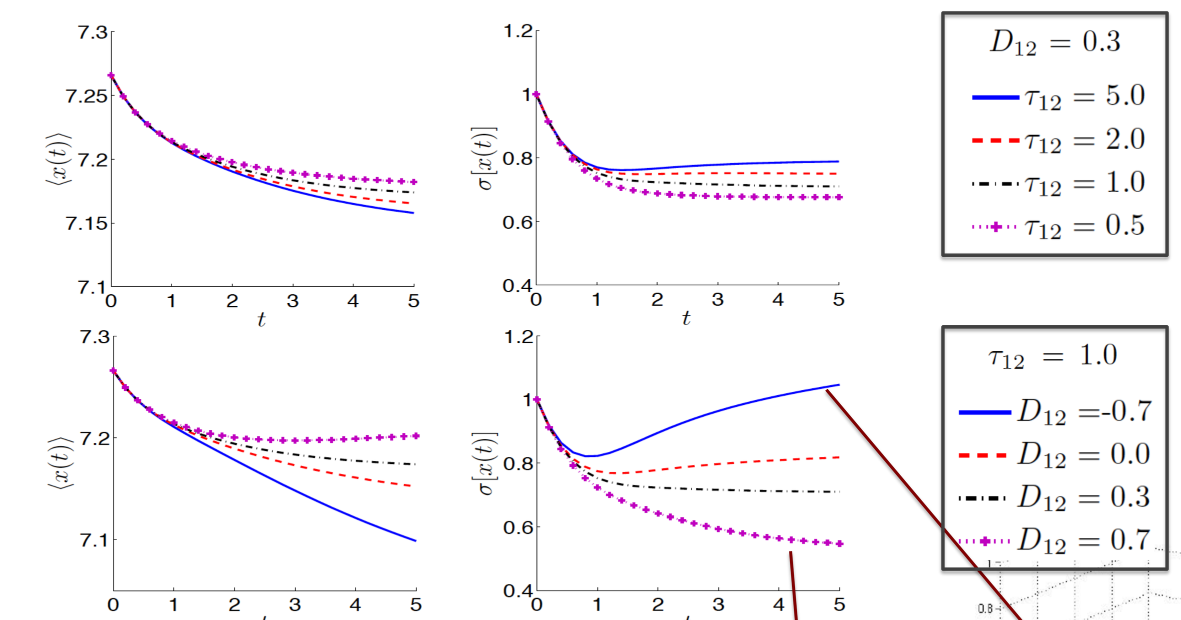
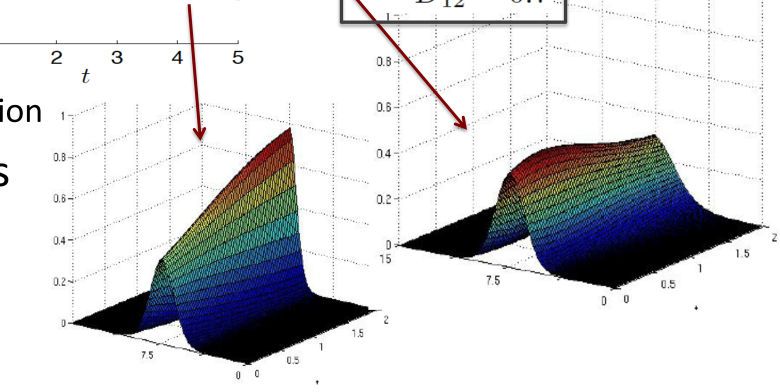


Figure 5. Mean and standard deviation of tumor concentration x(t).

➔The tumor cell concentration decays faster with larger variance when the cross correlation is stronger and has smaller magnitude.



Conclusions

Two difference methods, muKL and mcKL, have been proposed to represent multi-correlated non-stationary random processes.

• muKL method usually provides better accuracy and convergence rate, but it is computationally more expensive than mcKL.
• mcKL method yields scalable algorithms and explicit representation can be obtained. Also, it can be applied when each process are expanded in terms of random variables with different distribution.

These methods can be readily employed in stochastic simulation and we have demonstrated the importance of modeling the cross-correlation structure of the noise in a stochastic tumor model.

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