Abstract
We propose two different approaches generalizing the Karhunen-Loeve series expansion to model and simulate multi-correlated non-stationary stochastic processes. The muKL (multi-uncorrelated KL) method produces a series in terms an identical set of uncorrelated random variables, and mcKL (multi-correlated KL) relies on expansions in terms of correlated sets of random variables, both reflecting the cross-covariance structure of the processes. In particular, we study the accuracy and convergence rates of our series expansions and compare the results against other statistical techniques.

Introduction
Many stochastic systems of interest to engineering involve multiple random processes with mutual correlations, for instance, earthquake ground motions, acoustic propagation and multi-scale modeling of materials [1,2] (See Figure 1 and 2). The effective mathematical representation of such processes is the key element for the efficient stochastic simulations.

Over the years many different techniques have been developed for a single random process. A popular one employs series expansions in terms of random variables, including Karhunen-Loeve (KL) decompositions [3,4].

For stochastic process with following mean and covariance kernel 
\[ f(t) = (f(t;\omega), \quad C(t, s) = (f(t;\omega)f(s;\omega)) \]
the KL expansion can be written as
\[ f(t;\omega) = \overline{f(t)} + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \phi_i(t) \zeta_i(\omega) \]
where \( \lambda_i, \phi_i \) are eigenvalue and orthogonal eigenfunction of the covariance kernel, and \( \zeta_i(\omega) \) are independent uncorrelated random variables.

Methods
The effectiveness of KL expansion, for instance, optimal convergence in mean square error, is due to its bi-orthogonality, meaning that the eigenfunctions are orthogonal in \( L^2 \) and the random variables are uncorrelated. However, it restricts its application to single random process.

For multi-correlated random processes with following statistical structure,
\[ (f_1(t;\omega), f_j(t;\omega)) = (C_{ij}(t,\tau)) \quad 1 \leq i \leq n, \quad 1 \leq j \leq n \]
We propose two algorithms by relaxing one of the bi-orthogonality [5].

<table>
<thead>
<tr>
<th>multiple uncorrelated KL (muKL)</th>
<th>multiple correlated KL (mcKL)</th>
</tr>
</thead>
</table>
| Expansion with single set of uncorrelated random var. \( f(t;\omega) = \sum \sqrt{\lambda_1} \phi_1(t) \zeta_1(\omega) \) | Expansion with multiple set of correlated random var. \( f(t;\omega) = \sum \sqrt{\lambda_1} \phi_1(t) \zeta_1(\omega) \)

Results
1. Sample path : Both muKL and mcKL methods properly capture the correlated structure of multiple processes. See Figure 3.
2. Convergence : muKL method has better convergence in terms of mean square error. Table 1 shows the number of random variable to achieve less than 3% error in the eigenvalues.
3. Computational cost : mcKL is more efficient than muKL, due to its smaller size of eigenproblem and availability of some explicit results. See Figure 4.

Application
Tumor concentration with treatment modeled by random processes. 
\[ f(t;\omega) = C(t) + g(x_1(t;\omega) + f_2(t;\omega)) \]
where \( C(t;\omega) \) is the concentration of the tumor cell at time \( t \), 
\( f_2(t;\omega) \) factors that restrain the tumor cell (e.g., drugs, radiotherapy), 
where random processes are mutually correlated with the following structure, 
\[ \Phi(t;\omega) = C(t;\omega) + g(\xi(t;\omega)) \]
\[ \Phi(t;\omega) = \Phi(\omega) = \Phi(\omega) \]

Conclusions
Two different methods, muKL and mcKL, have been proposed to represent multi-correlated non-stationary random processes.

- • muKL method usually provides better accuracy and convergence rate, but it is computationally more expensive than mcKL.
  • mcKL method yields scalable algorithms and explicit representation can be obtained. Also, it can be applied when each process are expanded in terms of random variables with different distribution.

These methods can be readily employed in stochastic simulation and we have demonstrated the importance of modeling the cross-correlation structure of the noise in a stochastic tumor model.

References